

On higher index differential-algebraic equations in infinite dimensions

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Abstract. We consider initial value problems for differential-algebraic equations in a possibly infinite-dimensional Hilbert space. Assuming a growth condition for the associated operator pencil, we prove existence and uniqueness of solutions for arbitrary initial values in a distributional sense. Moreover, we construct a nested sequence of subspaces for initial values in order to obtain classical solutions.

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1. Introduction and main results

In this short note, we consider two solution concepts of differential-algebraic equations (DAEs) in infinite dimensions. For this, let E and A be bounded linear operators in some possibly infinite dimensional Hilbert space H .

We consider the implicit initial value problem

$$\begin{cases} Eu'(t) + Au(t) = 0, & t > 0, \\ u(0+) = u_0 \end{cases} \quad (*)$$

for some given $u_0 \in H$. In order to talk about a well-defined problem in $(*)$, we assume that the pair (E, A) is *regular*, that is,

$$\exists \nu \in \mathbb{R}: \mathbb{C}_{\operatorname{Re} > \nu} \subseteq \rho(E, A),$$

$$\exists C \geq 0, k \in \mathbb{N} \forall s \in \mathbb{C}_{\operatorname{Re} > \nu}: \|(sE + A)^{-1}\| \leq C|s|^k,$$

where

$$\rho(E, A) := \{s \in \mathbb{C}; (sE + A)^{-1} \in L(H)\}.$$

We note here that these two conditions are our replacements for regularity in finite dimensions. Indeed, for H finite-dimensional, (E, A) is called

regular, if $\det(sE + A) \neq 0$ for some $s \in \mathbb{C}$. Thus, $s \mapsto \det(sE + A)$ is a polynomial of degree at most $\dim H$, which is not identically zero. The growth condition is a consequence of the Weierstrass or Jordan normal form theorem valid for finite spatial dimensions, see e.g. [1, 2, 4]. The smallest possible $k \in \mathbb{N}$ occurring in the resolvent estimate is called the *index of* (E, A) :

$$\text{ind}(E, A) := \min\{k \in \mathbb{N}; \exists C \geq 0 \forall s \in \mathbb{C}_{\text{Re} > \nu} : \|(sE + A)^{-1}\| \leq C|s|^k\}.$$

We shall also define a sequence of (initial value) spaces associated with (E, A) :

$$\text{IV}_0 := H \text{ and } \text{IV}_{k+1} := \{x \in H; Ax \in E[\text{IV}_k]\} \quad (k \in \mathbb{N}).$$

A first observation is the following.

Proposition 1.1. *Let $k = \text{ind}(E, A)$ and assume that $E[\text{IV}_k] \subseteq H$ is closed. Then $\text{IV}_{k+1} = \text{IV}_{k+2}$.*

Since the sequence of spaces $(\text{IV}_k)_k$ is decreasing (see Lemma 3.1), Proposition 1.1 leads to the following question.

Problem 1.2. Assume that $E[\text{IV}_j] \subseteq H$ is closed for each $j \in \mathbb{N}$. Do we then have

$$\min\{k \in \mathbb{N}; \text{IV}_{k+1} = \text{IV}_{k+2}\} = \text{ind}(E, A)?$$

With the spaces $(\text{IV}_k)_k$ at hand, we can present the main theorem of this article.

Theorem 1.3. *Assume that $E[\text{IV}_{\text{ind}(E,A)}] \subseteq H$ is closed, $u_0 \in \text{IV}_{\text{ind}(E,A)+1}$. Then there exists a unique continuously differentiable function $u: \mathbb{R}_{>0} \rightarrow H$ with $u(0+) = u_0$ such that*

$$Eu'(t) + Au(t) = 0 \quad (t > 0).$$

With Proposition 1.1 and Theorem 1.3, it is possible to derive the following consequence.

Corollary 1.4. *Assume that $E[\text{IV}_j] \subseteq H$ is closed for each $j \in \mathbb{N}$, $u_0 \in H$. Then there exists a continuously differentiable function $u: \mathbb{R}_{>0} \rightarrow H$ with $u(0+) = u_0$ and*

$$Eu'(t) + Au(t) = 0 \quad (t > 0),$$

if, and only if, $u_0 \in \text{IV}_{\text{ind}(E,A)+1}$.

Corollary 1.4 suggests that the answer to Problem 1.2 is in the affirmative for H being finite-dimensional.

Also in our main result, there is room for improvement: In applications, it is easier to show that $R(E) \subseteq H$ is closed as the IV-spaces are not straightforward to compute. Thus, we ask whether the latter theorem can be improved in the following way.

Problem 1.5. Does $R(E) \subseteq H$ closed imply the closedness of $E[\text{IV}_{\text{ind}(E,A)}] \subseteq H$ or even closedness of $E[\text{IV}_j] \subseteq H$ for all $j \in \mathbb{N}$?

We shall briefly comment on the organization of this article. In the next section, we introduce the time-derivative operator in a suitably weighted vector-valued L_2 -space. This has been used intensively in the framework of so-called ‘evolutionary equations’, see [6]. With this notion, it is possible to obtain a *distributional* solution of $(*)$ such that the differential algebraic equation holds in an integrated sense, where the number of integrations needed corresponds to the index of the DAE. We conclude this article with the proofs of Proposition 1.1, Theorem 1.3, and Corollary 1.4. We emphasize that we do not employ any Weierstrass or Jordan normal theory in the proofs of our main results. We address the case of unbounded A to future research. The case of index 0 is discussed in [8], where also exponential stability and dichotomies are studied.

2. The time derivative and weak solutions of DAEs

Throughout this section, we assume that H is a Hilbert space and that $E, A \in L(H)$ with (E, A) regular. We start out with the definition of the space of (equivalence classes of) vector-valued L_2 functions: Let $\nu \in \mathbb{R}$. Then we set

$$L_{2,\nu}(\mathbb{R}; H) := \left\{ f : \mathbb{R} \rightarrow H ; f \text{ measurable, } \int_{\mathbb{R}} |f(t)|_H^2 \exp(-2\nu t) dt < \infty \right\},$$

see also [6, 3, 5]. Note that $L_{2,0}(\mathbb{R}; H) = L_2(\mathbb{R}; H)$. We define $H_\nu^1(\mathbb{R}; H)$ to be the (H -valued) Sobolev space of $L_{2,\nu}(\mathbb{R}; H)$ -functions with weak derivative representable as $L_{2,\nu}(\mathbb{R}; H)$ -function. With this, we can define the derivative operator

$$\partial_{0,\nu} : H_\nu^1(\mathbb{R}; H) \subseteq L_{2,\nu}(\mathbb{R}; H) \rightarrow L_{2,\nu}(\mathbb{R}; H), \phi \mapsto \phi'.$$

In the next theorem we recall some properties of the operator just defined. For this, we introduce the *Fourier–Laplace transformation* $\mathcal{L}_\nu : L_{2,\nu}(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ as being the unitary extension of

$$\mathcal{L}_\nu \phi(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(s) e^{-(it+\nu)s} ds \quad (\phi \in C_c(\mathbb{R}; H), t \in \mathbb{R}),$$

where $C_c(\mathbb{R}; H)$ denotes the space of compactly supported, continuous H -valued functions defined on \mathbb{R} . Moreover, let

$$\begin{aligned} \mathfrak{m} : \{f \in L_2(\mathbb{R}; H); (t \mapsto tf(t)) \in L_2(\mathbb{R}; H)\} &\subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H), \\ &f \mapsto (t \mapsto tf(t)) \end{aligned}$$

be the multiplication by the argument operator with maximal domain.

Theorem 2.1 ([3, Corollary 2.5]). *Let $\nu \in \mathbb{R}$. Then*

$$\partial_{0,\nu} = \mathcal{L}_\nu^*(\mathfrak{m} + \nu)\mathcal{L}_\nu.$$

Remark 2.2. A direct consequence of Theorem 2.1 is the continuous invertibility of $\partial_{0,\nu}$ if $\nu \neq 0$.

Corollary 2.3. *Let $\nu > 0$ be such that $\rho(E, A) \supseteq \mathbb{C}_{\text{Re} > \nu}$ and $\|(sE + A)^{-1}\| \leq C|s|^{\text{ind}(E, A)}$ for some $C \geq 0$ and all $s \in \mathbb{C}_{\text{Re} > \nu}$. Then*

$$\partial_{0, \nu}^{-k} (\partial_{0, \nu} E + A)^{-1} \in L(L_{2, \nu}(\mathbb{R}; H)),$$

where $k = \text{ind}(E, A)$. Moreover, $\partial_{0, \nu}^{-k} (\partial_{0, \nu} E + A)^{-1}$ is causal, i.e., for each $f \in L_{2, \nu}(\mathbb{R}; H)$ with $\text{spt } f \subseteq \mathbb{R}_{\geq a}$ for some $a \in \mathbb{R}$ it follows that

$$\text{spt } \partial_{0, \nu}^{-k} (\partial_{0, \nu} E + A)^{-1} f \subseteq \mathbb{R}_{\geq a}.$$

Proof. By Theorem 2.1 and the unitarity of \mathcal{L}_ν , we obtain that the first claim is equivalent to

$$(\text{im} + \nu)^{-k} ((\text{im} + \nu) E + A)^{-1} \in L(L_2(\mathbb{R}; H)),$$

which, in turn, would be implied by the fact that the function

$$t \mapsto (it + \nu)^{-k} ((it + \nu) E + A)^{-1}$$

belongs to $L^\infty(\mathbb{R}; L(H))$. This is, however, true by regularity of (E, A) . We now show the causality. As the operator $\partial_{0, \nu}^{-k} (\partial_{0, \nu} E + A)^{-1}$ commutes with translation in time, it suffices to prove the claim for $a = 0$. So let $f \in L_{2, \nu}(\mathbb{R}; H)$ with $\text{spt } f \subseteq \mathbb{R}_{\geq 0}$. By a Paley-Wiener type result (see e.g. [7, 19.2 Theorem]), the latter is equivalent to

$$(\mathbb{C}_{\text{Re} > \nu} \ni z \mapsto (\mathcal{L}_{\text{Re } z} f)(\text{Im } z)) \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \nu}; H),$$

where $\mathcal{H}^2(\mathbb{C}_{\text{Re} > \nu}; H)$ denotes the Hardy-space of H -valued functions on the half-plane $\mathbb{C}_{\text{Re} > \nu}$. As

$$\left(\mathcal{L}_{\text{Re } z} \partial_{0, \nu}^{-k} (\partial_{0, \nu} E + A)^{-1} f \right) (\text{Im } z) = z^{-k} (zE + A)^{-1} (\mathcal{L}_{\text{Re } z} f)(\text{Im } z)$$

for each $z \in \mathbb{C}_{\text{Re} > \nu}$, we infer that also

$$(\mathbb{C}_{\text{Re} > \nu} \ni z \mapsto \left(\mathcal{L}_{\text{Re } z} \partial_{0, \nu}^{-k} (\partial_{0, \nu} E + A)^{-1} f \right) (\text{Im } z)) \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \nu}; H),$$

due to the boundedness and analyticity of

$$\left(\mathbb{C}_{\text{Re} > \nu} \ni z \mapsto z^{-k} (zE + A)^{-1} \in L(H) \right).$$

This proves the claim. \square

Corollary 2.3 states a particular boundedness property for the solution operator associated with $(*)$. This can be made more precise by introducing a scale of extrapolation spaces associated with $\partial_{0, \nu}$.

Definition 2.4. Let $k \in \mathbb{N}$, $\nu > 0$. Then we define $H_\nu^k(\mathbb{R}; H) := D(\partial_{0, \nu}^k)$ endowed with the scalar product $\langle \phi, \psi \rangle_k := \langle \partial_{0, \nu}^k \phi, \partial_{0, \nu}^k \psi \rangle_0$. Quite similarly, we define $H_\nu^{-k}(\mathbb{R}; H)$ as the completion of $L_{2, \nu}(\mathbb{R}; H)$ with respect to $\langle \phi, \psi \rangle_{-k} := \langle \partial_{0, \nu}^{-k} \phi, \partial_{0, \nu}^{-k} \psi \rangle_0$.

We observe that the spaces $(H_\nu^k(\mathbb{R}; H))_{k \in \mathbb{Z}}$ are nested in the sense that $j_{k \rightarrow \ell} : H_\nu^k(\mathbb{R}; H) \hookrightarrow H_\nu^\ell(\mathbb{R}; H)$, $x \mapsto x$, whenever $k \geq \ell$.

Remark 2.5. The operator $\partial_{0,\nu}^\ell$ can be considered as a densely defined isometry from H^k to $H^{k-\ell}$ with dense range for all $k \in \mathbb{Z}$. The closure of this densely defined isometry will be given the same name. In this way, we can state the boundedness property of the solution operator in Corollary 2.3 equivalently as follows:

$$(\partial_{0,\nu}E + A)^{-1} \in L(L_{2,\nu}(\mathbb{R}; H), H_\nu^{-k}(\mathbb{R}; H)).$$

More generally, as $(\partial_{0,\nu}E + A)^{-1}$ and $\partial_{0,\nu}^{-1}$ commute, we obtain

$$(\partial_{0,\nu}E + A)^{-1} \in L(H_\nu^j(\mathbb{R}; H), H_\nu^{j-k}(\mathbb{R}; H))$$

for each $j \in \mathbb{Z}$.

Note that by the Sobolev embedding theorem (see e.g. [3, Lemma 5.2]) the δ -distribution of point evaluation at 0 is an element of $H_\nu^{-1}(\mathbb{R}; H)$; in fact it is the derivative of $\chi_{\mathbb{R}_{\geq 0}} \in L_{2,\nu}(\mathbb{R}; H) = H_\nu^0(\mathbb{R}; H)$. With these preparations at hand, we consider the following implementation of the initial value problem stated in (*): Let $u_0 \in H$. Find $u \in H_\nu^{-k}(\mathbb{R}; H)$ such that

$$(\partial_{0,\nu}E + A)u = \delta \cdot Eu_0. \quad (2.1)$$

Theorem 2.6. *Let (E, A) be regular. Then for all $u_0 \in H$ there exists a unique $u \in H_\nu^{-k}(\mathbb{R}; H)$ such that (2.1) holds. Moreover, we have*

$$u = \chi_{\mathbb{R}_{\geq 0}}u_0 - (\partial_{0,\nu}E + A)^{-1}\chi_{\mathbb{R}_{\geq 0}}Au_0$$

and

$$\text{spt } \partial_{0,\nu}^{-k}u \subseteq \mathbb{R}_{\geq 0}.$$

Proof. Note that the unique solution is given by

$$u = (\partial_{0,\nu}E + A)^{-1}\delta \cdot Eu_0 \in H_\nu^{-k-1}(\mathbb{R}; H).$$

Hence,

$$\begin{aligned} u - \chi_{\mathbb{R}_{\geq 0}}u_0 &= (\partial_{0,\nu}E + A)^{-1}(\delta \cdot Eu_0 - (\partial_{0,\nu}E + A)\chi_{\mathbb{R}_{\geq 0}}u_0) \\ &= -(\partial_{0,\nu}E + A)^{-1}\chi_{\mathbb{R}_{\geq 0}}Au_0, \end{aligned}$$

which shows the desired formula. Since $\chi_{\mathbb{R}_{\geq 0}}u_0 \in L_{2,\nu}(\mathbb{R}; H) \hookrightarrow H_\nu^{-k}(\mathbb{R}; H)$ and $(\partial_{0,\nu}E + A)^{-1}\chi_{\mathbb{R}_{\geq 0}}Au_0 \in H_\nu^{-k}(\mathbb{R}; H)$ by Corollary 2.3 we obtain the asserted regularity for u . The support statement follows from the causality statement in Corollary 2.3. \square

In the concluding section, we will discuss the spaces IV_k in connection to (E, A) and will prove the main results of this paper mentioned in the introduction.

3. Proofs of the main results and initial value spaces

Again, we assume that H is a Hilbert space, and that $E, A \in L(H)$ with (E, A) regular.

At first, we turn to the proof of Proposition 1.1. For this, we note some elementary consequences of the definition of IV_k and of regularity.

Lemma 3.1. (a) For all $k \in \mathbb{N}$, we have $\text{IV}_k \supseteq \text{IV}_{k+1}$.

(b) Let $s \in \mathbb{C} \cap \rho(E, A)$. Then

$$E(sE + A)^{-1}A = A(sE + A)^{-1}E.$$

(c) Let $k \in \mathbb{N}$, $x \in \text{IV}_k$. Then for all $s \in \mathbb{C} \cap \rho(E, A)$ we have

$$(sE + A)^{-1}Ex \in \text{IV}_{k+1}.$$

(d) Let $s \in \mathbb{C} \cap \rho(E, A) \setminus \{0\}$. Then

$$(sE + A)^{-1}E = \frac{1}{s} - \frac{1}{s}(sE + A)^{-1}A.$$

(e) Let $k \in \mathbb{N}$, $x \in \text{IV}_k$. Then for all $s \in \mathbb{C} \cap \rho(E, A) \setminus \{0\}$ we have

$$(sE + A)^{-1}Ex = \frac{1}{s}x + \sum_{\ell=1}^k \frac{1}{s^{\ell+1}}x_\ell + \frac{1}{s^{k+1}}(sE + A)^{-1}Aw.$$

for some $w \in H$, $x_1, \dots, x_k \in H$.

Proof. The proof of (a) is an induction argument. The claim is trivial for $k = 0$. For the inductive step, we see that the assertion follows using the induction hypothesis by

$$\text{IV}_{k+1} = A^{-1}[E[\text{IV}_k]] \supseteq A^{-1}[E[\text{IV}_{k+1}]] = \text{IV}_{k+2}.$$

Next, we prove (b). We compute

$$\begin{aligned} E(sE + A)^{-1}A &= E(sE + A)^{-1}(sE + A - sE) \\ &= E - E(sE + A)^{-1}sE \\ &= E - (sE + A - A)(sE + A)^{-1}E \\ &= A(sE + A)^{-1}E. \end{aligned}$$

We prove (c), by induction on k . For $k = 0$, we let $x \in \text{IV}_0 = H$ and put $y := (sE + A)^{-1}Ex$. Then, by (b), we get that

$$Ay = A(sE + A)^{-1}Ex = E(sE + A)^{-1}Ax \in R(E) = E[\text{IV}_0].$$

Hence, $y \in \text{IV}_1$. For the inductive step, we assume that the assertion holds for some $k \in \mathbb{N}$. Let $x \in \text{IV}_{k+1}$. We need to show that $y := (sE + A)^{-1}Ex \in \text{IV}_{k+2}$. For this, note that there exists $w \in \text{IV}_k$ such that $Ax = Ew$. In

particular, by the induction hypothesis, we have $(sE + A)^{-1}Ew \in \text{IV}_{k+1}$. Then we compute using (b) again,

$$\begin{aligned} Ay &= A(sE + A)^{-1}Ex \\ &= E(sE + A)^{-1}Ax \\ &= E(sE + A)^{-1}Ew \in E[\text{IV}_{k+1}]. \end{aligned}$$

Hence, $y \in \text{IV}_{k+2}$ and (c) is proved.

For (d), it suffices to observe

$$\begin{aligned} (sE + A)^{-1}E &= \frac{1}{s}(sE + A)^{-1}sE \\ &= \frac{1}{s}(sE + A)^{-1}(sE + A - A) \\ &= \frac{1}{s} - \frac{1}{s}(sE + A)^{-1}A. \end{aligned}$$

In order to prove part (e), we proceed by induction on $k \in \mathbb{N}$. The case $k = 0$ has been dealt with in part (d) by choosing $w = -x$. For the inductive step, we let $x \in \text{IV}_{k+1}$. By definition of IV_{k+1} , we find $y \in \text{IV}_k$ such that $Ax = Ey$. By induction hypothesis, we find $w \in H$ and $x_1, \dots, x_k \in H$ such that

$$(sE + A)^{-1}Ey = \frac{1}{s}y + \sum_{\ell=1}^k \frac{1}{s^{\ell+1}}x_\ell + \frac{1}{s^{k+1}}(sE + A)^{-1}Aw.$$

With this we compute using (d)

$$\begin{aligned} (sE + A)^{-1}Ex &= \frac{1}{s}x - \frac{1}{s}(sE + A)^{-1}Ax \\ &= \frac{1}{s}x - \frac{1}{s}(sE + A)^{-1}Ey \\ &= \frac{1}{s}x - \frac{1}{s} \left(\frac{1}{s}y + \sum_{\ell=1}^k \frac{1}{s^{\ell+1}}x_\ell + \frac{1}{s^{k+1}}(sE + A)^{-1}Aw \right) \\ &= \frac{1}{s}x + \sum_{\ell=1}^{k+1} \frac{1}{s^{\ell+1}}\tilde{x}_\ell + \frac{1}{s^{k+2}}(sE + A)^{-1}A\tilde{w}, \end{aligned}$$

with $\tilde{x}_1 = -y$, $\tilde{x}_\ell = -x_{\ell-1}$ for $\ell \geq 2$ and $\tilde{w} = -w$. □

With Lemma 3.1(a), we obtain the following reformulation of Proposition 1.1.

Proposition 3.2. *Assume that $E[\text{IV}_{\text{ind}(E,A)}] \subseteq H$ is closed. Then*

$$\text{IV}_{\text{ind}(E,A)+1} \subseteq \text{IV}_{\text{ind}(E,A)+2}.$$

Proof. Note that the closedness of $E[\text{IV}_{\text{ind}(E,A)}]$ implies the same for the space $\text{IV}_{\text{ind}(E,A)+1}$ since A is continuous. We set $k := \text{ind}(E, A)$. Let $x \in \text{IV}_{k+1}$. Then we need to find $y \in \text{IV}_{k+1}$ with $Ax = Ey$. By definition there exists $x_0 \in \text{IV}_k$ with the property $Ax = Ex_0$. For $n \in \mathbb{N}$ large enough we define $y_n := n(nE + A)^{-1}Ex_0$. Since, $x_0 \in \text{IV}_k$, we deduce with Lemma 3.1(c)

that $y_n \in \text{IV}_{k+1}$. Moreover, by Lemma 3.1(e), $(y_n)_n$ is bounded. Choosing a suitable subsequence for which we use the same name, we may assume that $(y_n)_n$ is weakly convergent to some $y \in H$. The closedness of IV_{k+1} implies $y \in \text{IV}_{k+1}$. Then using Lemma 3.1(e) we find $w \in H$ and $x_1, \dots, x_{k+1} \in H$ such that

$$(nE + A)^{-1} Ex_0 = \sum_{\ell=0}^k \frac{1}{n^{\ell+1}} x_\ell + \frac{1}{n^{k+1}} (nE + A)^{-1} Aw.$$

Hence, we obtain

$$\begin{aligned} Ey &= \text{w-lim}_{n \rightarrow \infty} Ey_n \\ &= \text{w-lim}_{n \rightarrow \infty} E(nE + A)^{-1} nEx_0 \\ &= \text{w-lim}_{n \rightarrow \infty} nE(nE + A)^{-1} Ax \\ &= \text{w-lim}_{n \rightarrow \infty} (nE + A - A)(nE + A)^{-1} Ax \\ &= Ax - \text{w-lim}_{n \rightarrow \infty} A(nE + A)^{-1} Ex_0 \\ &= Ax - A \text{w-lim}_{n \rightarrow \infty} \left(\sum_{\ell=0}^k \frac{1}{n^{\ell+1}} x_\ell + \frac{1}{n^{k+1}} (nE + A)^{-1} Aw \right) = Ax, \end{aligned}$$

which yields the assertion. \square

With an idea similar to the one in the proof of Proposition 1.1 (Proposition 3.2), it is possible to show that $E: \text{IV}_{k+1} \rightarrow E[\text{IV}_k]$ is an isomorphism if $k = \text{ind}(E, A)$ and $E[\text{IV}_k] \subseteq H$ is closed. We will need this result also in the proof of our main theorem.

Theorem 3.3. *Let (E, A) be regular and assume that $E[\text{IV}_k] \subseteq H$ is closed, $k = \text{ind}(E, A)$. Then*

$$E: \text{IV}_{k+1} \rightarrow E[\text{IV}_k], x \mapsto Ex$$

is a Banach space isomorphism.

Proof. Note that by the closed graph theorem, it suffices to show that the operator under consideration is one-to-one and onto. So, for proving injectivity, we let $x \in \text{IV}_{k+1}$ such that $Ex = 0$. By definition, there exists $y \in \text{IV}_k$ such that $Ey = Ax = Ax + nEx$ for all $n \in \mathbb{N}$. Hence, for $n \in \mathbb{N}$ large enough, we have $x = (nE + A)^{-1} Ey$. Thus, from $y \in \text{IV}_k$ we deduce with the help of Lemma 3.1(e) that there exist $w, x_1, \dots, x_k \in H$ such that

$$x = (nE + A)^{-1} Ey = \frac{1}{n} y + \sum_{\ell=1}^k \frac{1}{n^{\ell+1}} x_\ell + \frac{1}{n^{k+1}} (nE + A)^{-1} Aw \rightarrow 0 \quad (n \rightarrow \infty),$$

which shows $x = 0$.

Next, let $y \in E[\text{IV}_k]$. For large enough $n \in \mathbb{N}$ we put

$$w_n := (nE + A)^{-1} ny.$$

By Lemma 3.1(c), we obtain that $w_n \in \text{IV}_{k+1}$. Let $x \in \text{IV}_k$ with $Ex = y$. Then, using Lemma 3.1(e), we find $w, x_1, \dots, x_k \in H$ such that

$$\begin{aligned} w_n &= (nE + A)^{-1}ny \\ &= (nE + A)^{-1}nEx \\ &= x + \sum_{\ell=1}^k \frac{1}{n^\ell}x_\ell + \frac{1}{n^k}(nE + A)^{-1}Aw, \end{aligned}$$

proving the boundedness of $(w_n)_n$. Without loss of generality, we may assume that $(w_n)_n$ weakly converges to $z \in \text{IV}_{k+1} = A^{-1}[E[\text{IV}_k]]$. Hence,

$$\begin{aligned} Ez &= \text{w-lim}_{n \rightarrow \infty} Ew_n \\ &= \text{w-lim}_{n \rightarrow \infty} \frac{1}{n} (nE + A) w_n \\ &= \text{w-lim}_{n \rightarrow \infty} \frac{1}{n} (nE + A) (nE + A)^{-1}ny \\ &= y. \end{aligned} \quad \square$$

Next, we come to the proof of our main result Theorem 1.3, which we restate here for convenience.

Theorem 3.4. *Assume that $E[\text{IV}_{\text{ind}(E,A)}] \subseteq H$ is closed, $u_0 \in \text{IV}_{\text{ind}(E,A)+1}$. Then (2.1) has a unique continuously differentiable solution $u : \mathbb{R}_{>0} \rightarrow H$, satisfying $u(0+) = u_0$ and*

$$Eu'(t) + Au(t) = 0 \quad (t > 0). \quad (3.1)$$

Moreover, the solution coincides with the solution given in Theorem 2.6.

Proof. Let $u_0 \in \text{IV}_{\text{ind}(E,A)+1}$. We denote $\tilde{E} : \text{IV}_{k+1} \rightarrow E[\text{IV}_k], x \mapsto Ex$, where $k = \text{ind}(E, A)$. By Theorem 3.3, we have that \tilde{E} is an isomorphism. For $t > 0$, we define

$$u(t) := \exp\left(-t\tilde{E}^{-1}A\right)u_0.$$

Then $u(0+) = u_0$. Moreover, $u(t)$ is well-defined. Indeed, if $u_0 \in \text{IV}_{k+1}$ then $Au_0 \in E[\text{IV}_k]$. Hence, $\tilde{E}^{-1}Au_0 \in \text{IV}_{k+1}$ is well-defined. Since $E[\text{IV}_k]$ is closed, and A is continuous, we infer that IV_{k+1} is a Hilbert space. Thus, we deduce that $u : \mathbb{R}_{>0} \rightarrow \text{IV}_{k+1}$ is continuously differentiable. In particular, we obtain

$$\text{IV}_{k+1} \ni u'(t) = -\tilde{E}^{-1}Au(t).$$

If we apply \tilde{E} to both sides of the equality, we obtain (3.1). If $u : \mathbb{R}_{>0} \rightarrow H$ is a continuously differentiable solution of (3.1) with $u(0+) = u_0$, we infer that $u \in L_{2,\nu}(\mathbb{R}; H)$ for some $\nu > 0$ large enough, where we extend u to $\mathbb{R}_{<0}$ by zero. Hence,

$$\partial_{0,\nu}Eu + Au = E\partial_{0,\nu}u + Au = Eu' + Au + \delta \cdot Eu(0+) = \delta \cdot Eu_0,$$

where we have used that u is differentiable on $\mathbb{R}_{<0} \cup \mathbb{R}_{>0}$ and jumps at 0. Thus, u is the solution given in Theorem 2.6, from which we also derive the uniqueness. \square

We conclude with a comment on the proof of Corollary 1.4.

Remark 3.5. We note that the condition $u_0 \in \text{IV}_{\text{ind}(E,A)+1}$ arises naturally if we assume that IV_j is closed for each $j \in \mathbb{N}$. Indeed, if $u : \mathbb{R}_{>0} \rightarrow H$ is a continuously differentiable solution of (3.1), we infer that

$$Au(t) = -Eu'(t) \quad (t > 0)$$

and thus $u(t) \in \text{IV}_1$ for $t > 0$. Since IV_1 is closed, we derive $u'(t) \in \text{IV}_1$ and hence, inductively $u(t) \in \bigcap_{j \in \mathbb{N}} \text{IV}_j$ for each $t > 0$. Since $\bigcap_{j \in \mathbb{N}} \text{IV}_j = \text{IV}_{\text{ind}(E,A)+1}$ by Proposition 3.2, we get

$$u_0 = u(0+) \in \text{IV}_{\text{ind}(E,A)+1}.$$

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